Lower Bounds for Laplacian and Fractional Laplacian Eigenvalues

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ABSTRACT: In this paper, we investigate eigenvalues of Laplacian on a bounded domain in an n-dimensional Euclidean space and obtain a sharper lower bound for the sum of its eigenvalues, which gives an improvement of results due to A. D. Melas [15]. On the other hand, for the case of fractional Laplacian $(-\Delta)^{\alpha/2}|_D$, where $\alpha \in (0,2]$, we obtain a sharper lower bound for the sum of its eigenvalues, which gives an improvement of results due to S.Y. Yolcu and T. Yolcu [23].

1   Introduction

Let $D \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary ∂D in an n-dimensional Euclidean space $\mathbb{R}^n$. Let λ_i be the i-th eigenvalue of the fixed membrane problem:

$$\begin{cases} \Delta u + \lambda u = 0, & in \ D, \\ u = 0, & on \ \partial D, \end{cases} \tag{1.1}$$

where Δ is the Laplacian in $\mathbb{R}^n$. It is well known that the spectrum of this eigenvalue problem is real and discrete:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to +\infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity. If we use the notations $Vol(D)$ and ω_n to denote the volume of D and the volume of the unit ball in $\mathbb{R}^n$, respectively, then Weyl's asymptotic formula asserts that the eigenvalues of the fixed membrane problem (1.1) satisfy the following formula:

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to +\infty. \tag{1.2}$$

From the above asymptotic formula, it follows directly that

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to +\infty. \tag{1.3}$$

Pólya [17] proved that

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \cdots, \tag{1.4}$$

if D is a tiling domain in $\mathbb{R}^n$. Furthermore, he put forward the following:

Conjecture of Pólya. *If D is a bounded domain in $\mathbb{R}^n$, then the k-th eigenvalue λ_k of the fixed membrane problem satisfies*

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \cdots. \tag{1.5}$$

On the Conjecture of Pólya, Berezin [2] and Lieb [13] gave a partial solution. In particular, Li and Yau [13] proved the Berezin-Li-Yau inequality as follows:

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \cdots. \tag{1.6}$$

The formula (1.3) shows that the result of Li and Yau is sharp in the sense of average. From this inequality (1.6), one can derive

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \cdots, \tag{1.7}$$

which gives a partial solution for the conjecture of Pólya with a factor $\dfrac{n}{n+2}$. We prefer to call this inequality (1.6) as Berezin-Li-Yau inequality instead of Li-Yau inequality because (1.6) can be obtained by a Legendre transform of an earlier result by Berezin [2] as it is mentioned [14]. Recently, improvements to the Berezin-Li-Yau inequality given by (1.6) for the fixed membrane problem have appeared, for example see [10, 15, 20]. In particular, A.D.Melas [15] has improved the estimate (1.6) to the following:

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n Vol(D))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{1}{24(n+2)} \frac{Vol(D)}{Ine(D)}, \quad \text{for } k = 1, 2, \cdots, \tag{1.8}$$

where

$$Ine(D) =: \min_{a \in \mathbb{R}^n} \int_D |x - a|^2 dx$$

is called *the moment of inertia* of D. After a translation of the origin, we can assume that the center of mass is the origin and

$$Ine(D) = \int_D |x|^2 dx.$$

By taking a value nearby the extreme point of the function $f(\tau)$ (given by (**??**)), we add one term of lower order of $k^{-\frac{2}{n}}$ to its right hand side, which means that we obtain a sharper result than (1.8). In fact, we prove the following:

Theorem 1.1. *Let D be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Assume that $\lambda_i, i = 1, 2, \cdots,$ is the i-th eigenvalue of the eigenvalue problem (1.1). Then the sum of its eigenvalues satisfies*

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{nk^{\frac{2}{n}}}{n+2}\omega_n^{-\frac{2}{n}}(2\pi)^2 Vol(D)^{-\frac{2}{n}} + \frac{1}{24(n+2)}\frac{Vol(D)}{Ine(D)}$$

$$+ \frac{nk^{-\frac{2}{n}}}{2304(n+2)^2}\omega_n^{\frac{2}{n}}(2\pi)^{-2}\left(\frac{Vol(D)}{Ine(D)}\right)^2 Vol(D)^{\frac{2}{n}}. \tag{1.9}$$

Furthermore, we consider the fractional Laplacian operators restricted to D, and denote them by $(-\Delta)^{\alpha/2}|_D$, where $\alpha \in (0, 2]$. This fractional Laplacian can be defined by

$$(-\Delta)^{\alpha/2}u(x) =: \mathbf{P.V.}\int_{\mathbb{R}^n}\frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy,$$

where **P.V.** denotes the principal value and $u : \mathbb{R}^n \to \mathbb{R}$. Define the characteristic function $\chi_D : t \mapsto \chi_D(t)$ by

$$\chi_D(t) = \begin{cases} 1, & x \in D, \\ 0, & x \in \mathbb{R}^n\backslash D, \end{cases}$$

then the special pseudo-differential operator can be represented as the Fourier transform of the function u [11, 19], namely

$$(-\Delta)^{\alpha/2}|_D u := \mathscr{F}^{-1}[|\xi|^\alpha \mathscr{F}[u\chi_D]],$$

where $\mathscr{F}[u]$ denotes the Fourier transform of a function $u : \mathbb{R}^n \to \mathbb{R}$:

$$\mathscr{F}[u](\xi) = \widehat{u}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)dx.$$

It is well known that the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ can be considered as the infinitesimal generator of the symmetric α-stable process [3–6, 23]. Suppose that a stochastic process $\mathcal{X}_t$ has stationary independent increments and its transition density (i.e., convolution kernel) $p^\alpha(t, x, y) = p^\alpha(t, x - y)$, $t > 0$, $x, y \in \mathbb{R}^n$ is determined by the following Fourier transform

$$\text{Exp}(-t|\xi|^\alpha) = \int_{\mathbb{R}^n}e^{i\xi\cdot y}p^\alpha(t, y)dy, \quad t > 0, \quad \xi \in \mathbb{R}^n,$$

then we can say that the process $\mathcal{X}_t$ is an n-dimensional *symmetric α-stable process* with order $\alpha \in (0, 2]$ in $\mathbb{R}^n$(also see [4, 5, 23]).

Remark 1.1. *Given $\alpha = 1$, $\mathcal{X}_t$ is the Cauchy process in $\mathbb{R}^n$ whose transition densities are given by the Cauchy distribution (Poisson kernel)*

$$p^1(t, x, y) = \frac{c_n t}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}}, \quad t > 0, \quad x, y \in \mathbb{R}^n,$$

where

$$c_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}} = \frac{1}{\sqrt{\pi}\omega_n},$$

is the semiclassical constant that appears in the Weyl estimate for the eigenvalues of the Laplacian.

Remark 1.2. *Given $\alpha = 2$, $\mathcal{X}_t$ is just the usual n-dimensional Brownian motion $\mathcal{B}_t$ but running at twice the speed, which is equivalent to say that, when $\alpha = 2$, we have $\mathcal{X}_t = \mathcal{B}_{2t}$ and*

$$p^2(t,x,y) = \frac{1}{(4\pi t)^{n/2}}\mathrm{Exp}\left[\frac{-|x-y|^2}{4t}\right], \quad t > 0, \quad x, y \in \mathbb{R}^n.$$

Let Λ_j^α and u_j^α denote the j-th eigenvalue and the corresponding normalized eigenvector of $(-\Delta)^{\alpha/2}|_\Omega$, respectively. Eigenvalues Λ_j^α (including multiplicities) satisfy

$$0 < \Lambda_1^{(\alpha)} \le \Lambda_2^{(\alpha)} \le \Lambda_3^{(\alpha)} \le \cdots \to +\infty.$$

For the case of $\alpha = 1$, E. Harrell and S. Y. Yolcu gave an analogue of the Berezin-Li-Yau type inequality for the eigenvalues of the Klein-Gordon operators $\mathscr{H}_{0,D} := \sqrt{-\Delta}$ restricted to D in [9]:

$$\frac{1}{k}\sum_{j=1}^k \Lambda_j^{(\alpha)} \ge \frac{n}{n+1}\left(\frac{2\pi}{(\omega_n Vol(D))^{\frac{1}{n}}}\right)k^{\frac{1}{n}}. \tag{1.10}$$

Very recently, S.Y.Yolcu [22] has improved the estimate (1.10) to the following:

$$\frac{1}{k}\sum_{j=1}^k \Lambda_j^{(\alpha)} \ge \frac{n\widetilde{C}_n}{n+1}Vol(D)^{-\frac{1}{n}}k^{\frac{1}{n}} + \widetilde{M}_n\frac{Vol(D)^{1+\frac{1}{n}}}{Ine(D)}k^{-1/n}, \tag{1.11}$$

where $\widetilde{C}_n = \frac{2\pi}{(\omega_n)^{\frac{1}{n}}}$ and the constant $\widetilde{M}_n$ depends only on the dimension n. Moreover, for any $\alpha \in (0,2]$, S.Y.Yolcu and T.Yolcu [23] generalized (1.11) as follows:

$$\frac{1}{k}\sum_{j=1}^k \Lambda_j^{(\alpha)} \ge \frac{n}{n+\alpha}\left(\frac{2\pi}{(\omega_n Vol(D))^{\frac{1}{n}}}\right)^\alpha k^{\frac{\alpha}{n}}. \tag{1.12}$$

Furthermore, S.Y.Yolcu and T.Yolcu [23] refined the Berezin-Li-Yau inequality in the case of fractional Laplacian $(-\Delta)^\alpha|_D$ restricted to D:

$$\begin{aligned}
\frac{1}{k}\sum_j^k \Lambda_j^{(\alpha)} &\ge \frac{n}{n+\alpha}\left(\frac{2\pi}{(\omega_n Vol(D))^{\frac{1}{n}}}\right)^\alpha k^{\frac{\alpha}{n}} \\
&+ \frac{\ell}{4(n+\alpha)}\frac{(2\pi)^{\alpha-2}}{(\omega_n Vol(D))^{\frac{\alpha-2}{n}}}\frac{Vol(D)}{Ine(D)}k^{\frac{\alpha-2}{n}},
\end{aligned} \tag{1.13}$$

where ℓ is given by

$$\ell = \min\left\{\frac{\alpha}{12}, \frac{4\alpha n\pi^2}{(2n+2-\alpha)\omega_n^{\frac{4}{n}}}\right\}.$$

Remark 1.3. *In fact, by a direct calculation, one can check the following inequality:*

$$\frac{\alpha}{12} \leq \frac{4\alpha n \pi^2}{(2n + 2 - \alpha)\omega_n^{\frac{4}{n}}},$$

which implies

$$\frac{\ell}{4(n+\alpha)} \frac{(2\pi)^{\alpha-2}}{(\omega_n Vol(D))^{\frac{\alpha-2}{n}}} \frac{Vol(D)}{Ine(D)} k^{\frac{\alpha-2}{n}} = \frac{\alpha}{48(n+\alpha)} \frac{(2\pi)^{\alpha-2}}{(\omega_n Vol(D))^{\frac{\alpha-2}{n}}} \frac{Vol(D)}{Ine(D)} k^{\frac{\alpha-2}{n}}.$$

The another main purpose of this paper is to provide a refinement of the Berezin-Li-Yau type estimate. In other word, we have proved the following:

Theorem 1.2. *Let D be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Assume that $\Lambda_i^{(\alpha)}, i = 1, 2, \cdots,$ is the i-th eigenvalue of the fractional Laplacian $(-\Delta)^{\alpha/2}|_D$. Then, the sum of its eigenvalues satisfies*

$$\begin{aligned}
\frac{1}{k}\sum_{j=1}^k \Lambda_j^{(\alpha)} \geq\ & \frac{n}{n+\alpha} \frac{(2\pi)^\alpha}{(\omega_n Vol(D))^{\frac{\alpha}{n}}} k^{\frac{\alpha}{n}} \\
& + \frac{\alpha}{48(n+\alpha)} \frac{(2\pi)^{\alpha-2}}{(\omega_n Vol(D))^{\frac{\alpha-2}{n}}} \frac{Vol(D)}{Ine(D)} k^{\frac{\alpha-2}{n}} \\
& + \frac{\alpha(n+\alpha-2)^2}{\mathcal{C}(n)n(n+\alpha)^2} \frac{(2\pi)^{\alpha-4}}{(\omega_n Vol(D))^{\frac{\alpha-4}{n}}} \left(\frac{Vol(D)}{Ine(D)}\right)^2 k^{\frac{\alpha-4}{n}},
\end{aligned} \tag{1.14}$$

where

$$\mathcal{C}(n) = \begin{cases} 4608, & \text{when } n \geq 4, \\ 6144, & \text{when } n = 2 \text{ or } n = 3. \end{cases}$$

In particular, the sum of its eigenvalues satisfies

$$\begin{aligned}
\frac{1}{k}\sum_{j=1}^k \Lambda_j^{(2)} \geq\ & \frac{nk^{\frac{2}{n}}}{n+2}\omega_n^{-\frac{2}{n}}(2\pi)^2 Vol(D)^{-\frac{2}{n}} + \frac{1}{24(n+2)}\frac{Vol(D)}{Ine(D)} \\
& + \frac{nk^{-\frac{2}{n}}}{2304(n+2)^2}\omega_n^{\frac{2}{n}}(2\pi)^{-2}\left(\frac{Vol(D)}{Ine(D)}\right)^2 Vol(D)^{\frac{2}{n}},
\end{aligned} \tag{1.15}$$

when $\alpha = 2$.

Remark 1.4. *Observing Theorem 1.2, it is not difficult to see that the coefficients (with respect to $k^{\frac{\alpha-2}{n}}$) of the second terms in (1.14) are equal to that of (1.13). In other word, we can claim that the inequalities (1.14) are sharper than (1.13) since the coefficients (with respect to $k^{\frac{\alpha-4}{n}}$) of the third terms in (1.14) are positive.*

By using Theorem 1.2, we can give an analogue of the Berezin-Li-Yau type inequality for the eigenvalues of the Klein-Gordon operators $\mathscr{H}_{0,D}$ restricted to the bounded domain D:

Corollary 1.1. *Let D be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Assume that $\Lambda_i, i = 1, 2, \cdots$, is the i-th eigenvalue of the Klein-Gordon operators $\mathscr{H}_{0,D}$. Then, the sum of its eigenvalues satisfies*

$$
\begin{aligned}
\frac{1}{k}\sum_{j=1}^{k}\Lambda_j \geq \ & \frac{n}{n+1}\frac{2\pi}{(\omega_n Vol(D))^{\frac{1}{n}}}k^{\frac{1}{n}} \\
& + \frac{1}{48(n+1)}\frac{(2\pi)^{-1}}{(\omega_n Vol(D))^{-\frac{1}{n}}}\frac{Vol(D)}{Ine(D)}k^{-\frac{1}{n}} \\
& + \frac{(n-1)^2}{\mathcal{C}(n)n(n+1)^2}\frac{(2\pi)^{-3}}{(\omega_n Vol(D))^{-\frac{3}{n}}}\left(\frac{Vol(D)}{Ine(D)}\right)^2 k^{-\frac{3}{n}},
\end{aligned}
\tag{1.16}
$$

where

$$
\mathcal{C}(n) = \begin{cases} 4608, & \text{when} \quad n \geq 4, \\ 6144, & \text{when} \quad n = 2 \quad \text{or} \quad n = 3. \end{cases}
$$

2 A Key Lemma

In order to prove the following Lemma 2.3 , we need the following lemmas given by S.Y.Yolcu and T.Yolcu in [23]:

Lemma 2.1. *Suppose that $\varsigma : [0,\infty) \to [0,1]$ such that*

$$
0 \leq \varsigma(s) \leq 1 \text{ and } \int_0^\infty \varsigma(s)ds = 1.
$$

Then, there exists $\epsilon \geq 0$ such that

$$
\int_\epsilon^{\epsilon+1} s^d ds = \int_0^\infty s^d \varsigma(s)ds.
$$

Moreover, we have

$$
\int_\epsilon^{\epsilon+1} s^{d+\alpha} ds \leq \int_0^\infty s^{d+\alpha}\varsigma(s)ds.
$$

Lemma 2.2. *For $s > 0$, $\tau > 0$, $2 \leq b \in \mathbb{N}$, $0 < \alpha \leq 2$, we have the following inequality:*

$$
s^{b+\alpha} \geq \frac{b+\alpha}{b}s^b\tau^\alpha - \frac{\alpha}{b}\tau^{b+\alpha} + \frac{\alpha}{b}\tau^{b+\alpha-2}(s-\tau)^2.
$$

In light of Lemma 2.1 and Lemma 2.2, we obtain the following result which will play important roles in the proof of Theorem 1.1 and Theorem 1.2.

Lemma 2.3. *Let $b(\geq 2)$ be a positive real number and $\mu(> 0)$ be defined by (2.13). If $\psi : [0, +\infty) \to [0, +\infty)$ is a decreasing function such that*

$$
-\mu \leq \psi'(s) \leq 0
$$

and

$$A := \int_0^\infty s^{b-1}\psi(s)ds > 0,$$

then, we have

$$\int_0^\infty s^{b+\alpha-1}\psi(s)ds \geq \frac{1}{b+\alpha}(bA)^{\frac{b+\alpha}{b}}\psi(0)^{-\frac{\alpha}{b}}$$
$$+ \frac{\alpha}{12b(b+\alpha)\mu^2}(bA)^{\frac{b+\alpha-2}{b}}\psi(0)^{\frac{2b-\alpha+2}{b}} \qquad (2.1)$$
$$+ \frac{\alpha(b+\alpha-2)^2}{288b^2(b+\alpha)^2\mu^4}(bA)^{\frac{b+\alpha-4}{b}}\psi(0)^{\frac{4b-\alpha+4}{b}},$$

when $b \geq 4$; we have

$$\int_0^\infty s^{b+\alpha-1}\psi(s)ds \geq \frac{1}{b+\alpha}(bA)^{\frac{b+\alpha}{b}}\psi(0)^{-\frac{\alpha}{b}}$$
$$+ \frac{\alpha}{12b(b+\alpha)\mu^2}(bA)^{\frac{b+\alpha-2)}{b}}\psi(0)^{\frac{2b-\alpha+2}{b}} \qquad (2.2)$$
$$+ \frac{\alpha(b+\alpha-2)^2}{384b^2(b+\alpha)^2\mu^4}(bA)^{\frac{b+\alpha-4}{b}}\psi(0)^{\frac{4b-\alpha+4}{b}},$$

when $2 \leq b < 4$. In particular, the inequality (2.1) holds when $\alpha = 2$ and $b \geq 2$.

Proof. If we consider the following function

$$\varrho(t) = \frac{\psi\left(\frac{\psi(0)}{\mu}t\right)}{\psi(0)},$$

then it is not difficult to see that $\varrho(0) = 1$ and $-1 \leq \varrho'(t) \leq 0$. Without loss of generality, we can assume

$$\psi(0) = 1 \text{ and } \mu = 1.$$

Define

$$E_\alpha := \int_0^\infty s^{b+\alpha-1}\psi(s)ds.$$

One can assume that $E_\alpha < \infty$, otherwise there is nothing to prove. By the assumption, we can conclude that

$$\lim_{s\to\infty} s^{b+\alpha-1}\psi(s) = 0.$$

Putting $h(s) = -\psi'(s)$ for any $s \geq 0$, we get

$$0 \leq h(s) \leq 1 \text{ and } \int_0^\infty h(s)ds = \psi(0) = 1.$$

By making use of integration by parts, one can get

$$\int_0^\infty s^b h(s)ds = b\int_0^\infty s^{b-1}\psi(s)ds = bA,$$

and

$$\int_0^\infty s^{b+\alpha} h(s)ds \le (b+\alpha)E_\alpha,$$

since $\psi(s) > 0$. By Lemma 2.1, one can infer that there exists an $\epsilon \ge 0$ such that

$$\int_\epsilon^{\epsilon+1} s^b ds = \int_0^\infty s^b h(s)ds = bA, \tag{2.3}$$

and

$$\int_\epsilon^{\epsilon+1} s^{b+\alpha} ds \le \int_0^\infty s^{b+\alpha} h(s)ds \le (b+\alpha)E_\alpha. \tag{2.4}$$

Let

$$\Theta(s) = bs^{b+\alpha} - (b+\alpha)\tau^\alpha s^b + \alpha\tau^{b+\alpha} - \alpha\tau^{b+\alpha-2}(s-\tau)^2,$$

then, by Lemma 2.2, we have $\Theta(s) \ge 0$. Integrating the function $\Theta(s)$ from ϵ to $\epsilon + 1$, we deduce from (2.3) and (2.4), for any $\tau > 0$,

$$b(b+\alpha)E_\alpha - (b+\alpha)\tau^\alpha bA + \alpha\tau^{b+\alpha} \ge \frac{\alpha}{12}\tau^{b+\alpha-2}. \tag{2.5}$$

Define

$$f(\tau) := (b+\alpha)\tau^\alpha bA - \alpha\tau^{b+\alpha} + \frac{\alpha}{12}\tau^{b+\alpha-2}, \tag{2.6}$$

then we can obtain from (2.5) that, for any $\tau > 0$,

$$E_\alpha = \int_0^\infty s^{b+\alpha-1}\psi(s)ds \ge \frac{f(\tau)}{b(b+\alpha)}.$$

Taking

$$\tau = (bA)^{\frac{1}{b}}\left(1 + \frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^{\frac{1}{b}},$$

and substituting it into (2.6), we obtain

$$\begin{aligned}
f(\tau) = (bA)^{\frac{b+\alpha}{b}}\left(b - \frac{\alpha(b+\alpha-2)}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)\left(1 + \frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^{\frac{\alpha}{b}} \\
+ \frac{\alpha}{12}(bA)^{\frac{b+\alpha-2}{b}}\left(1 + \frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^{\frac{b+\alpha-2}{b}}.
\end{aligned} \tag{2.7}$$

By using the Taylor formula, one has for $t > 0$

$$\begin{aligned}
(1+t)^{\frac{\alpha}{b}} \ge 1 + \frac{\alpha}{b}t + \frac{\alpha(\alpha-b)}{2b^2}t^2 + \frac{\alpha(\alpha-b)(\alpha-2b)}{6b^3}t^3 \\
+ \frac{\alpha(\alpha-b)(\alpha-2b)(\alpha-3b)}{24b^4}t^4,
\end{aligned}$$

and

$$(1+t)^{\frac{b+\alpha-2}{b}} \geq 1 + \frac{b+\alpha-2}{b}t + \frac{(b+\alpha-2)(\alpha-2)}{2b^2}t^2$$
$$+ \frac{(b+\alpha-2)(\alpha-2)(\alpha-2-b)}{6b^3}t^3$$
$$+ \frac{(b+\alpha-2)(\alpha-2)(\alpha-2-b)(\alpha-2-2b)}{24b^4}t^4.$$

Putting

$$t = \frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}} > 0,$$

one has $b - \alpha t > 0$, $\tau = (bA)^{\frac{1}{b}}(1+t)^{\frac{1}{b}}$,

$$\left(b - \frac{\alpha(b+\alpha-2)}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)\left(1 + \frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^{\frac{\alpha}{b}}$$
$$= (b-\alpha t)(1+t)^{\frac{\alpha}{b}}$$
$$\geq (b-\alpha t)\left[1 + \frac{\alpha}{b}t + \frac{\alpha(\alpha-b)}{2b^2}t^2 + \frac{\alpha(\alpha-b)(\alpha-2b)}{6b^3}t^3\right.$$
$$\left. + \frac{\alpha(\alpha-b)(\alpha-2b)(\alpha-3b)}{24b^4}t^4\right]$$
$$= b - \frac{\alpha(\alpha+b)}{2b}t^2 - \frac{\alpha(\alpha-b)(\alpha+b)}{3b^2}t^3 - \frac{\alpha(\alpha-b)(\alpha-2b)(\alpha+b)}{8b^3}t^4$$
$$- \frac{\alpha^2(\alpha-b)(\alpha-2b)(\alpha-3b)}{24b^4}t^5 \qquad (2.8)$$
$$= b - \frac{\alpha(\alpha+b)}{2b}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^2$$
$$- \frac{\alpha(\alpha-b)(\alpha+b)}{3b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^3$$
$$- \frac{\alpha(\alpha-b)(\alpha-2b)(\alpha+b)}{8b^3}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^4$$
$$- \frac{\alpha^2(\alpha-b)(\alpha-2b)(\alpha-3b)}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^5,$$

and

$$
\left(1 + \frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^{\frac{b+\alpha-2}{b}}
$$

$$
= (1+t)^{\frac{b+\alpha-2}{b}}
$$

$$
\geq 1 + \frac{b+\alpha-2}{b}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)
$$

$$
+ \frac{(b+\alpha-2)(\alpha-2)}{2b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^2 \tag{2.9}
$$

$$
+ \frac{(b+\alpha-2)(\alpha-2)(\alpha-2-b)}{6b^3}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^3
$$

$$
+ \frac{(b+\alpha-2)(\alpha-2)(\alpha-2-b)(\alpha-2-2b)}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^4.
$$

Therefore, we obtain from (2.8) and (2.9)

$$
f(\tau) = (b+\alpha)\tau^\alpha bA - \alpha\tau^{b+\alpha} + \frac{\alpha}{12}\tau^{b+\alpha-2}
$$

$$
\geq (bA)^{\frac{b+\alpha}{b}}\left[b - \frac{\alpha(\alpha+b)}{2b}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^2\right.
$$

$$
- \frac{\alpha(\alpha-b)(\alpha+b)}{3b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^3
$$

$$
- \frac{\alpha(\alpha-b)(\alpha-2b)(\alpha+b)}{8b^3}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^4
$$

$$
\left.- \frac{\alpha^2(\alpha-b)(\alpha-2b)(\alpha-3b)}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^5\right] \tag{2.10}
$$

$$
+ \frac{\alpha}{12}(bA)^{\frac{b+\alpha-2}{b}}\left[1 + \frac{b+\alpha-2}{b}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)\right.
$$

$$
+ \frac{(b+\alpha-2)(\alpha-2)}{2b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^2
$$

$$
+ \frac{(b+\alpha-2)(\alpha-2)(\alpha-2-b)}{6b^3}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^3
$$

$$
\left.+ \frac{(b+\alpha-2)(\alpha-2)(\alpha-2-b)(\alpha-2-2b)}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}(bA)^{-\frac{2}{b}}\right)^4\right]
$$

$$
= b(bA)^{\frac{b+\alpha}{b}} + \frac{\alpha}{12}(bA)^{\frac{b+\alpha-2}{b}} + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
$$

where

$$\mathcal{I}_1 = \frac{\alpha(b+\alpha-2)^2}{288b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}}, \tag{2.11}$$

$$\mathcal{I}_2 = \frac{\alpha(b+\alpha-2)(\alpha+2b-6)}{72b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}}$$
$$+ \frac{\alpha(b+\alpha-2)[\alpha^2+(5b-16)\alpha+(-6b^2+8b+16)]}{288b^3} \tag{2.12}$$
$$\times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^3 (bA)^{\frac{b+\alpha-8}{b}},$$

$$\mathcal{I}_3 = \frac{\alpha\gamma}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^5 (bA)^{\frac{b+\alpha-10}{b}},$$

and

$$\gamma = (\alpha-2)(\alpha-2-b)(\alpha-2-2b)(b+\alpha) - \alpha(\alpha-b)(\alpha-2b)(\alpha-3b).$$

Noticing that

$$-\alpha(\alpha-b)(\alpha-2b)(\alpha-3b) \geq 0,$$

we have

$$\gamma \geq (\alpha-2)(\alpha-2-b)(\alpha-2-2b)(b+\alpha).$$

Define

$$\beta := (\alpha-2)(\alpha-2-b)(\alpha-2-2b)(b+\alpha),$$

then we have $\beta \leq 0$ and $\gamma \geq \beta$. Therefore, we have

$$\mathcal{I}_3 \geq \frac{\alpha\beta}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^5 (bA)^{\frac{b+\alpha-10}{b}}, \tag{2.13}$$

Next, we consider two cases:

Case 1: b ≥ 4. When $b \geq 4$, for any $\alpha \in (0,2]$, we can infer

$$\alpha^2 + (5b-16)\alpha + (-6b^2+8b+16) \leq (5b-16)\alpha + (-6b^2+8b+20)$$
$$= -6b^2 + (8+5\alpha)b + 20 - 16\alpha$$
$$\leq -24b + (8+10)b + 20 \tag{2.14}$$
$$\leq 0.$$

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Since $(bA)^{\frac{2}{b}} \geq \frac{1}{(b+1)^{\frac{2}{b}}} \geq \frac{1}{3}$ (see [8]), one can deduce from (2.12) and (2.14)

$$
\begin{aligned}
\mathcal{I}_2 &\geq \frac{\alpha(b+\alpha-2)(\alpha+2b-6)}{72b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}} \\
&\quad + \frac{\alpha(b+\alpha-2)[\alpha^2+(5b-16)\alpha+(-6b^2+8b+16)]}{1152b^3} \\
&\quad \times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}} \\
&= \frac{\alpha(b+\alpha-2)\big[16b(\alpha+2b-6)+\alpha^2+(5b-16)\alpha+(-6b^2+8b+16)\big]}{1152b^3} \\
&\quad \times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}} \\
&= \frac{\alpha(b+\alpha-2)\big[26b^2+(-88+21\alpha)b+(\alpha^2-16\alpha+16)\big]}{1152b^3} \\
&\quad \times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}}.
\end{aligned} \tag{2.15}
$$

On the other hand, we have

$$
\mathcal{I}_3 \geq \frac{\alpha\beta}{4608b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}},
$$

since $\beta \leq 0$ and $(bA)^{\frac{2}{b}} \geq \frac{1}{(b+1)^{\frac{2}{b}}} \geq \frac{1}{3}$. Therefore, the estimate of the lower bound of $\mathcal{I}_2 + \mathcal{I}_3$ can be given by

$$
\begin{aligned}
\mathcal{I}_2 + \mathcal{I}_3 &\geq \left\{\frac{\alpha(b+\alpha-2)\big[26b^2+(-88+21\alpha)b+(\alpha^2-16\alpha+16)\big]}{1152b^3} + \frac{\alpha\beta}{4608b^4}\right\} \\
&\quad \times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}} \\
&= \frac{\alpha\{4b(b+\alpha-2)[26b^2+(-88+21\alpha)b+(\alpha^2-16\alpha+16)]+\beta\}}{4608b^4} \\
&\quad \times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}}.
\end{aligned}
$$

Next, we will verify the following inequality

$$
4b(b+\alpha-2)[26b^2+(-88+21\alpha)b+(\alpha^2-16\alpha+16)]+\beta \geq 0. \tag{2.16}
$$

Indeed, since $0 < \alpha \le 2$ and $b \ge 4$, we have

$$
\begin{aligned}
&4b(b+\alpha-2)[26b^2 + (-88+21\alpha)b + (\alpha^2 - 16\alpha + 16)] + \beta \\
&= 4b(b+\alpha-2)[26b^2 + (-88+21\alpha)b + (\alpha^2 - 16\alpha + 16)] \\
&\quad + (\alpha-2)(\alpha-2-b)(\alpha-2-2b)(b+\alpha) \\
&\ge 8b[26b^2 + (-88+21\alpha)b + (\alpha^2 - 16\alpha + 16)] \\
&\quad - |(\alpha-2)(\alpha-2-b)(\alpha-2-2b)(b+\alpha)| \\
&\ge 8b[26b^2 + (-88+21\alpha)b + (\alpha^2 - 16\alpha + 16)] - 2|(b+2)(2b+2)(b+2)| \\
&\ge 8b[26b^2 - 88b + (\alpha^2 - 8\alpha)] - 2(b+2)(2b+2)(b+2) \\
&\ge 8b(26b^2 - 92b) - 2(b+2)(2b+2)(b+2) \\
&= 204b^3 - 756b^2 - 32b - 16 \\
&\ge 60b^2 - 32b - 16 \\
&\ge 28b - 16 \\
&\ge 0.
\end{aligned}
\tag{2.17}
$$

Thus, it is not difficult to see that the inequality (2.16) follows from (2.17), which implies

$$\mathcal{I}_2 + \mathcal{I}_3 \ge 0.$$

Therefore, when $b \ge 4$, we have

$$f(\tau) \ge b(bA)^{\frac{b+\alpha}{b}} + \frac{\alpha}{12}(bA)^{\frac{b+\alpha-2}{b}} + \frac{\alpha(b+\alpha-2)^2}{288b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}}.$$

Case 2: $2 \le \mathbf{b} < 4$. Uniting the equations (2.11), (2.12) and (2.13), we obtain the following equation

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \ge{}& \frac{\alpha(b+\alpha-2)^2}{288b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}} \\
&+ \frac{\alpha(b+\alpha-2)(\alpha+2b-6)}{72b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}} \\
&+ \frac{\alpha(b+\alpha-2)[\alpha^2 + (5b-16)\alpha + (-6b^2 + 8b + 16)]}{288b^3} \\
&\times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^3 (bA)^{\frac{b+\alpha-8}{b}} \\
&+ \frac{\alpha\beta}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^5 (bA)^{\frac{b+\alpha-10}{b}}
\end{aligned}
$$

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$$= \frac{\alpha(b+\alpha-2)^2}{384b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}} + \frac{\alpha(b+\alpha-2)^2}{1152b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}}$$

$$+ \frac{\alpha(b+\alpha-2)\nu_1}{72b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2(bA)^{\frac{b+\alpha-6}{b}}$$

$$+ \frac{\alpha(b+\alpha-2)\nu_2}{288b^3}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^3(bA)^{\frac{b+\alpha-8}{b}}$$

$$+ \frac{\alpha\beta}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^5(bA)^{\frac{b+\alpha-10}{b}},$$

where

$$\nu_1 := (\alpha + 2b - 6),$$

and

$$\nu_2 := \alpha^2 + (5b-16)\alpha + (-6b^2 + 8b + 16).$$

Suppose $\nu_1 \leq 0$ and $\nu_2 \leq 0$, then we have

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \geq \frac{\alpha(b+\alpha-2)^2}{384b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}} + \frac{\alpha(b+\alpha-2)}{96b}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}}$$

$$+ \frac{\alpha(b+\alpha-2)(\alpha+2b-6)}{288b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}}$$

$$+ \frac{\alpha(b+\alpha-2)[\alpha^2 + (5b-16)\alpha + (-6b^2 + 8b + 16)]}{4608b^3}$$

$$\times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}} \tag{2.18}$$

$$+ \frac{\alpha\beta}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^5(bA)^{\frac{b+\alpha-10}{b}}$$

$$= \frac{\alpha(b+\alpha-2)^2}{384b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}} + \frac{\alpha(b+\alpha-2)}{96b}\mathcal{I}_4\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}}$$

$$+ \frac{\alpha\beta}{24b^4}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^5(bA)^{\frac{b+\alpha-10}{b}},$$

where

$$\mathcal{I}_4 = 1 + \frac{\alpha + 2b - 6}{3b} + \frac{\alpha^2 + (5b-16)\alpha + (-6b^2 + 8b + 16)}{48b^2}.$$

Noticing that $0 < \alpha \le 2$ and $2 \le b < 4$, we have

$$
\begin{aligned}
\mathcal{I}_4 &= \frac{48b^2 + 16b(\alpha + 2b - 6) + \alpha^2 + (5b - 16)\alpha + (-6b^2 + 8b + 16)}{48b^2} \\
&= \frac{74b^2 + (21\alpha - 88)b + (\alpha^2 - 16\alpha + 16)}{48b^2} \\
&\ge \frac{60b + 21\alpha b + (\alpha^2 - 8\alpha)}{48b^2} \\
&= \frac{60b + (21b - 8)\alpha + \alpha^2}{48b^2} \\
&\ge \frac{60b}{48b^2} = \frac{5}{4b}.
\end{aligned}
\tag{2.19}
$$

Therefore, we derive from (2.18) and (2.19)

$$
\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 &\ge \frac{\alpha(b + \alpha - 2)^2}{384b(b + \alpha)}(bA)^{\frac{b+\alpha-4}{b}} + \frac{5\alpha(b + \alpha - 2)}{384b^2}\left(\frac{b + \alpha - 2}{12(b + \alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}} \\
&\quad + \frac{\alpha\beta}{24b^4}\left(\frac{b + \alpha - 2}{12(b + \alpha)}\right)^5 (bA)^{\frac{b+\alpha-10}{b}} \\
&\ge \frac{\alpha(b + \alpha - 2)^2}{384b(b + \alpha)}(bA)^{\frac{b+\alpha-4}{b}} + \frac{5\alpha(b + \alpha - 2)}{384b^2}\left(\frac{b + \alpha - 2}{12(b + \alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}} \\
&\quad + \frac{\alpha(b + \alpha - 2)\beta}{18432(b + \alpha)b^4}\left(\frac{b + \alpha - 2}{12(b + \alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}} \\
&\ge \frac{\alpha(b + \alpha - 2)^2}{384b(b + \alpha)}(bA)^{\frac{b+\alpha-4}{b}} \\
&\quad + \frac{\alpha(b + \alpha - 2)[240b^2(b + \alpha) + \beta]}{18432(b + \alpha)b^4}\left(\frac{b + \alpha - 2}{12(b + \alpha)}\right)(bA)^{\frac{b+\alpha-4}{b}},
\end{aligned}
$$

since $(bA)^{\frac{2}{b}} \ge \frac{1}{(b+1)^{\frac{2}{b}}} \ge \frac{1}{3}$. We define a function $\mathcal{K}(b)$ by letting

$$
\begin{aligned}
\mathcal{K}(b) &:= 240b^2(b + \alpha) + \beta \\
&= 240b^2(b + \alpha) + (\alpha - 2)(\alpha - 2 - b)(\alpha - 2 - 2b)(b + \alpha),
\end{aligned}
$$

where $b \in [2, 4)$. After a direct calculation, we have

$$
\begin{aligned}
\mathcal{K}(b) &\ge 240b^2(b + \alpha) - |(\alpha - 2)(\alpha - 2 - b)(\alpha - 2 - 2b)(b + \alpha)| \\
&\ge 240b^2(b + \alpha) - |2(2 + b)(2 + 2b)(b + 2)| \\
&\ge 240b^2(b + \alpha) - 2(2b)(3b)(2b) \\
&\ge 216b^3 + 240\alpha b^2 > 0,
\end{aligned}
$$

which implies

$$
\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \ge \frac{\alpha(b + \alpha - 2)^2}{384b(b + \alpha)}(bA)^{\frac{b+\alpha-4}{b}}.
$$

For the other cases (i.e., $\nu_1 \le 0$ and $\nu_2 > 0$; $\nu_1 > 0$ and $\nu_2 \le 0$; or $\nu_1 > 0$ and $\nu_2 > 0$), we can also derive by using the same method that

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \ge \frac{\alpha(b+\alpha-2)^2}{384b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}}.$$

Therefore, when $2 \le b \le 4$, we have

$$f(\tau) \ge b(bA)^{\frac{b+\alpha}{b}} + \frac{\alpha}{12}(bA)^{\frac{b+\alpha-2}{b}} + \frac{\alpha(b+\alpha-2)^2}{384b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}}.$$

In particular, we can consider the case that $\alpha = 2$. Noticing that $\beta = 0$ when $\alpha = 2$ and $b \ge 2$, we can claim that $\mathcal{I}_3 \ge 0$. Therefore, when $\alpha = 2$ and $b \ge 2$, one can deduce

$$
\begin{aligned}
\mathcal{I}_2 + \mathcal{I}_3 \ge\ & \frac{\alpha(b+\alpha-2)(\alpha+2b-6)}{72b^2}\left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^2 (bA)^{\frac{b+\alpha-6}{b}} \\
& + \frac{\alpha(b+\alpha-2)[\alpha^2+(5b-16)\alpha+(-6b^2+8b+16)]}{288b^3} \\
& \times \left(\frac{b+\alpha-2}{12(b+\alpha)}\right)^3 (bA)^{\frac{b+\alpha-8}{b}} \\
=\ & \frac{b(b-2)}{18b^2}\left(\frac{b}{12(b+2)}\right)^2 (bA)^{\frac{b-4}{b}} + \frac{-b^2+3b-2}{24b^2}\left(\frac{b}{12(b+2)}\right)^3 (bA)^{\frac{b-6}{b}} \\
\ge\ & \frac{b(b-2)}{18b^2}\left(\frac{b}{12(b+2)}\right)^2 (bA)^{\frac{b-4}{b}} + \frac{-b^2+3b-2}{96b^2}\left(\frac{b}{12(b+2)}\right)^2 (bA)^{\frac{b-4}{b}} \\
=\ & \frac{13b^2-23b-6}{288b^2}\left(\frac{b}{12(b+2)}\right)^2 (bA)^{\frac{b-4}{b}} \\
\ge\ & \frac{26b-23b-6}{288b^2}\left(\frac{b}{12(b+2)}\right)^2 (bA)^{\frac{b-4}{b}} \\
\ge\ & 0,
\end{aligned}
$$

which implies

$$f(\tau) \ge b(bA)^{\frac{b+\alpha}{b}} + \frac{\alpha}{12}(bA)^{\frac{b+\alpha-2}{b}} + \frac{\alpha(b+\alpha-2)^2}{288b(b+\alpha)}(bA)^{\frac{b+\alpha-4}{b}}.$$

This completes the proof of the Lemma 2.3.

$\square$

3 Proofs of Theorem 1.1 and Theorem 1.2

In this section, we will prove the Theorem 1.1 and Theorem 1.2 by using the key lemma given in section 2 (i.e., Lemma 2.3).

We suppose that $D \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$, and then its *symmetric rearrangement* D^* is the open ball with the same volume as D,

$$D^* = \left\{ x \in \mathbb{R}^n \, \middle| \, |x| < \left(\frac{Vol(D)}{\omega_n} \right)^{\frac{1}{n}} \right\}.$$

By using a symmetric rearrangement of D, one can obtain

$$Ine(D) = \int_D |x|^2 dx \geq \int_{D^*} |x|^2 dx = \frac{n}{n+2} Vol(D) \left(\frac{Vol(D)}{\omega_n} \right)^{\frac{2}{n}}. \qquad (3.1)$$

For the case of fractional Laplace operator, let $u_j^{(\alpha)}$ be an orthonormal eigenfunction corresponding to the eigenvalue $\Lambda_j^{(\alpha)}$. Namely, $u_j^{(\alpha)}$ satisfies

$$\begin{cases} (-\Delta)^{\alpha/2} u_j^{(\alpha)} = \Lambda^{(\alpha)} u_j^{(\alpha)}, & \text{in } D, \\ \int_D u_i^{(\alpha)}(x) u_j^{(\alpha)}(x) dx = \delta_{ij}, & \text{for any } i, j, \end{cases}$$

where $0 < \alpha \leq 2$. On the other hand, for the case of Laplace operator, we let v_j be an orthonormal eigenfunction corresponding to the eigenvalue λ_j. Namely, v_j satisfies

$$\begin{cases} \Delta v_j + \lambda_j v_j = 0, & \text{in } D, \\ v = 0, & \text{on } \partial D, \\ \int_D v_i(x) v_j(x) dx = \delta_{ij}, & \text{for any } i, j. \end{cases}$$

Thus, both $\{u_j^{(\alpha)}\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ form an orthonormal basis of $L^2(D)$. Define the functions $\varphi_j^{(\alpha)}$ and η_j by

$$\varphi_j^{(\alpha)}(x) = \begin{cases} u_j^{(\alpha)}(x), & x \in D, \\ 0, & x \in \mathbb{R}^n \backslash D, \end{cases}$$

and

$$\eta_j(x) = \begin{cases} v_j(x), & x \in D, \\ 0, & x \in \mathbb{R}^n \backslash D, \end{cases}$$

respectively. Denote by $\widehat{\eta_j}(\xi)$ and $\widehat{\varphi_j^{(\alpha)}}(\xi)$ the Fourier transforms of $\eta_j(\xi)$ and $\varphi_j^{(\alpha)}(\xi)$, then, for any $\xi \in \mathbb{R}^n$, we have

$$\widehat{\varphi_j^{(\alpha)}}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_j^{(\alpha)}(x) e^{i\langle x, \xi \rangle} dx = (2\pi)^{-n/2} \int_D u_j^{(\alpha)}(x) e^{i\langle x, \xi \rangle} dx,$$

and

$$\widehat{\eta_j}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \eta_j(x) e^{i\langle x, \xi \rangle} dx = (2\pi)^{-n/2} \int_D v_j(x) e^{i\langle x, \xi \rangle} dx.$$

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From the Plancherel formula, we have

$$\int_{\mathbb{R}^n} \widehat{\varphi_i^{(\alpha)}}(x)\widehat{\varphi_j^{(\alpha)}}(x)dx = \int_{\mathbb{R}^n} \widehat{\eta_i}(x)\widehat{\eta_j}(x)dx = \delta_{ij},$$

for any i, j. Since $\{u_j^{(\alpha)}\}_{j=1}^\infty$ and $\{v_j\}_{j=1}^\infty$ are orthonormal basises in $L^2(D)$, the Bessel inequality implies that

$$\sum_{j=1}^k |\widehat{\varphi_j^{(\alpha)}}(\xi)|^2 \le (2\pi)^{-n/2} \int_D |e^{i\langle x,\xi\rangle}|^2 dx = (2\pi)^{-n/2} Vol(D), \tag{3.2}$$

and

$$\sum_{j=1}^k |\widehat{\eta_j}(\xi)|^2 \le (2\pi)^{-n/2} \int_D |e^{i\langle x,\xi\rangle}|^2 dx = (2\pi)^{-n/2} Vol(D). \tag{3.3}$$

For fractional Laplace operator, we observe that

$$\begin{aligned}
\Lambda_j^{(\alpha)} &= \int_{\mathbb{R}^n} u_j^{(\alpha)}(\xi) \cdot (-\Delta)^{\alpha/2}|_\Omega u_j^{(\alpha)}(\xi)d\xi \\
&= \int_{\mathbb{R}^n} u_j^{(\alpha)}(\xi) \cdot \mathscr{F}^{-1}[|\xi|^\alpha \mathscr{F}[u_j^{(\alpha)}(\xi)]]d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{u_j^{(\alpha)}}(\xi)|^2 d\xi,
\end{aligned} \tag{3.4}$$

since the support of $u_j^{(\alpha)}$ is D (see [23]). On the meanwhile, for the case of Laplace operator, we have (see [12, 15])

$$\lambda_j = \int_{\mathbb{R}^n} |\xi|^2 |\widehat{v_j}(\xi)|^2 d\xi. \tag{3.5}$$

Since

$$\nabla\widehat{\varphi_j^{(\alpha)}}(\xi) = (2\pi)^{-n/2} \int_\Omega ixu_j^{(\alpha)}(x)e^{i\langle x,\xi\rangle}dx,$$

and

$$\nabla\widehat{\eta_j}(\xi) = (2\pi)^{-n/2} \int_\Omega ixv_j(x)e^{i\langle x,\xi\rangle}dx,$$

we obtain

$$\sum_{j=1}^k |\nabla\widehat{\varphi_j^{(\alpha)}}(\xi)|^2 = \sum_{j=1}^k |\nabla\widehat{\eta_j}(\xi)|^2 = (2\pi)^{-n} \int_\Omega |ixe^{i\langle x,\xi\rangle}|^2 dx = (2\pi)^{-n} Ine(D). \tag{3.6}$$

Putting

$$f^{(\alpha)}(\xi) := \sum_{j=1}^k |\widehat{\varphi_j^{(\alpha)}}(\xi)|^2,$$

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and

$$f(\xi) := \sum_{j=1}^{k} |\widehat{\eta_j}(\xi)|^2,$$

one derives from (3.2) and (3.3) that $0 \leq f^{(\alpha)}(\xi) \leq (2\pi)^{-n} Vol(D)$ and $0 \leq f(\xi) \leq (2\pi)^{-n} Vol(D)$, it follows from (3.6) and the Cauchy-Schwarz inequality that

$$|\nabla f^{(\alpha)}(\xi)| \leq 2 \left(\sum_{j=1}^{k} |\widehat{\varphi_j^{(\alpha)}}(\xi)|^2 \right)^{1/2} \left(\sum_{j=1}^{k} |\nabla \widehat{\varphi_j^{(\alpha)}}(\xi)|^2 \right)^{1/2}$$

$$\leq 2(2\pi)^{-n} \sqrt{Ine(D)Vol(D)},$$

and

$$|\nabla f(\xi)| \leq 2 \left(\sum_{j=1}^{k} |\widehat{\eta_j}(\xi)|^2 \right)^{1/2} \left(\sum_{j=1}^{k} |\nabla \widehat{\eta_j}(\xi)|^2 \right)^{1/2}$$

$$\leq 2(2\pi)^{-n} \sqrt{Ine(D)Vol(D)},$$

for every $\xi \in \mathbb{R}^n$. Furthermore, by using (3.4) and (3.5), we have

$$\sum_{j=1}^{k} \Lambda_j^{(\alpha)} = \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{u_j^{(\alpha)}}(\xi)|^2 d\xi = \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{\varphi_j^{(\alpha)}}(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^n} |\xi|^\alpha f^{(\alpha)}(\xi) d\xi, \tag{3.7}$$

and

$$\sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\xi|^2 |\widehat{v_j}(\xi)|^2 d\xi = \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\xi|^2 |\widehat{\eta_j}(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^n} |\xi|^2 f(\xi) d\xi. \tag{3.8}$$

From the Parseval's identity, we derive

$$\int_{\mathbb{R}^n} f^{(\alpha)}(\xi) d\xi = \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\widehat{\varphi_j^{(\alpha)}}(x)|^2 dx = \sum_{j=1}^{k} \int_{D} |\widehat{u_j^{(\alpha)}}(x)|^2 dx$$
$$= \sum_{j=1}^{k} \int_{D} |u_j^{(\alpha)}(x)|^2 dx = k. \tag{3.9}$$

Similarly, we have [7, 15]

$$\int_{\mathbb{R}^n} f(\xi) d\xi = \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\widehat{\eta_j}(x)|^2 dx = \sum_{j=1}^{k} \int_{D} |\widehat{v_j}(x)|^2 dx$$
$$= \sum_{j=1}^{k} \int_{D} |v_j(x)|^2 dx = k. \tag{3.10}$$

Let h be a nonnegative bounded continuous function on D and h^* is its symmetric decreasing rearrangement, then we have (see [1,7])

$$\int_{\mathbb{R}^n} h(x)dx = \int_{\mathbb{R}^n} h^*(x)dx = n\omega_n \int_0^\infty s^{n-1}g(s)ds \tag{3.11}$$

and

$$\int_{\mathbb{R}^n} |x|^\alpha h(x)dx \geq \int_{\mathbb{R}^n} |x|^\alpha h^*(x)dx = n\omega_n \int_0^\infty s^{n+\alpha-1}g(s)ds, \tag{3.12}$$

where $\alpha \in (0,2]$ and $g(|x|) = h^*(x)$. Putting $\delta := \sup|\nabla h|$, then we can obtain

$$-\delta \leq g'(s) \leq 0 \tag{3.13}$$

for almost every s. More detail information on symmetric decreasing rearrangements will be found in [1,7,18].

To be brief, we will drop the superscript α to denote $f^{(\alpha)}$ by f_1 and let $f_2 = f$. Assume that f_i^* is the symmetric decreasing rearrangement of f_i $(i = 1,2)$, according to (3.9), (3.10) and (3.11), we have

$$k = \int_{\mathbb{R}^n} f_i(\xi)d\xi = \int_{\mathbb{R}^n} f_i^*(\xi)d\xi = n\omega_n \int_0^\infty s^{n-1}\phi_i(s)ds, \tag{3.14}$$

where $\phi_i(x) = f_i^*(|x|)$ and $i = 1,2$.

Applying the symmetric decreasing rearrangement to f_i, and noting that

$$\delta_i \leq 2(2\pi)^{-n}\sqrt{Ine(\Omega)Vol(\Omega)} := \sigma, \tag{3.15}$$

where $\delta_i = \sup|\nabla f_i|$, we obtain from (3.13)

$$-\sigma \leq -\delta_i \leq \phi_i'(s) \leq 0,$$

where $i = 1,2$. By(3.1), we have

$$\sigma \geq 2(2\pi)^{-n}(\frac{n}{n+2})^{\frac{1}{2}}\omega_n^{-\frac{1}{n}}Vol(D)^{\frac{n+1}{n}} \geq (2\pi)^{-n}\omega_n^{-\frac{1}{n}}Vol(D)^{\frac{n+1}{n}},$$

since $n \geq 2$. Moreover, by using (3.7), (3.8) and (3.12), we have

$$\begin{aligned}
\sum_{j=1}^k \Lambda_j^{(\alpha)} &= \int_{\mathbb{R}^n} |\xi|^\alpha f^{(\alpha)}(\xi)d\xi = \int_{\mathbb{R}^n} |\xi|^\alpha f_1(\xi)d\xi \\
&\geq \int_{\mathbb{R}^n} |\xi|^\alpha f_1^*(\xi)d\xi = n\omega_n \int_0^\infty s^{n+\alpha-1}\phi_1(\xi)d\xi,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\sum_{j=1}^k \lambda_j &= \int_{\mathbb{R}^n} |\xi|^2 f(\xi)d\xi = \int_{\mathbb{R}^n} |\xi|^2 f_2(\xi)d\xi \\
&\geq \int_{\mathbb{R}^n} |\xi|^2 f_2^*(\xi)d\xi = n\omega_n \int_0^\infty s^{n+1}\phi_2(\xi)d\xi.
\end{aligned} \tag{3.17}$$

Proof of Theorem 1.1. In order to apply Lemma 2.3, from (3.14), (3.15) and the definition of A, we take

$$b = n, \quad \psi(s) = \phi_2(s), \quad A = \frac{k}{n\omega_n}, \quad \text{and} \quad \mu = \sigma = 2(2\pi)^{-n}\sqrt{Vol(D)Ine(D)}.$$

Therefore, we can obtain from Lemma 2.3 and (3.17) that

$$\sum_{j=1}^{k} \lambda_j = n\omega_n E_2$$

$$\geq \frac{n\omega_n}{n(n+2)}\left[n(nA)^{\frac{n+2}{n}}\phi_2(0)^{-\frac{2}{n}} + \frac{nA\phi_2(0)^2}{6\sigma^2} + \frac{n(nA)^{\frac{n-2}{n}}}{144(n+2)\sigma^4}\phi_2(0)^{\frac{4n+2}{n}}\right]$$

$$= \frac{\omega_n}{n+2}\left[n\left(\frac{k}{\omega_n}\right)^{\frac{n+2}{n}}t^{-\frac{2}{n}} + \frac{\frac{k}{\omega_n}t^2}{6\sigma^2} + \frac{n(\frac{k}{\omega_n})^{\frac{n-2}{n}}}{144(n+2)\sigma^4}t^{\frac{4n+2}{n}}\right]$$

$$= \frac{n}{n+2}\omega_n^{-\frac{2}{n}}k^{\frac{n+2}{n}}t^{-\frac{2}{n}} + \frac{kt^2}{6(n+2)\sigma^2} + \frac{n\omega_n^{\frac{2}{n}}k^{\frac{n-2}{n}}}{144(n+2)^2\sigma^4}t^{\frac{4n+2}{n}},$$

where $t = \phi_2(0)$. Let

$$F(t) = \frac{n}{n+2}\omega_n^{-\frac{2}{n}}k^{\frac{n+2}{n}}t^{-\frac{2}{n}} + \frac{kt^2}{6(n+2)\sigma^2} + \frac{n\omega_n^{\frac{2}{n}}k^{\frac{n-2}{n}}}{144(n+2)^2\sigma^4}t^{\frac{4n+2}{n}},$$

then one can has

$$F'(t) = -\frac{2}{n+2}\omega_n^{-\frac{2}{n}}k^{\frac{n+2}{n}}t^{-\frac{n+2}{n}} + \frac{kt}{3(n+2)\sigma^2} + \frac{4n+2}{144(n+2)^2\sigma^4}\omega_n^{\frac{2}{n}}k^{\frac{n-2}{n}}t^{\frac{3n+2}{n}}.$$

Since $F'(t)$ is increasing on $(0, (2\pi)^{-n}Vol(D)]$, then it is easy to see that $F(t)$ is decreasing on $(0, (2\pi)^{-n}Vol(D)]$ if $F'((2\pi)^{-n}Vol(D)) < 0$. Indeed,

$$F'((2\pi)^{-n}Vol(D)) \leq -\frac{2}{n+2}\omega_n^{-\frac{2}{n}}k^{\frac{n+2}{n}}((2\pi)^{-n}Vol(D))^{-\frac{n+2}{n}}$$

$$+ \frac{k((2\pi)^{-n}Vol(D))}{3(n+2)\left[(2\pi)^{-n}\omega_n^{-\frac{1}{n}}Vol(D)^{\frac{n+1}{n}}\right]^2}$$

$$+ \frac{(4n+2)\omega_n^{\frac{2}{n}}k^{\frac{n-2}{n}}((2\pi)^{-n}Vol(D))^{\frac{3n+2}{n}}}{144(n+2)^2\left[(2\pi)^{-n}\omega_n^{-\frac{1}{n}}Vol(D)^{\frac{n+1}{n}}\right]^4}$$

$$= -\frac{2}{n+2}(2\pi)^{n+2}\omega_n^{-\frac{2}{n}}k^{\frac{n+2}{n}}Vol(D)^{-\frac{n+2}{n}}$$

$$+ \frac{1}{3(n+2)}(2\pi)^n\omega_n^{\frac{2}{n}}kVol(D)^{-\frac{n+2}{n}}$$

$$+ \frac{4n+2}{144(n+2)^2}(2\pi)^{n-2}\omega_n^{\frac{6}{n}}k^{\frac{n-2}{n}}Vol(D)^{-\frac{n+2}{n}}$$

$$= \frac{(2\pi)^nk}{n+2}\omega_n^{\frac{2}{n}}Vol(D)^{-\frac{n+2}{n}}\mathcal{J},$$

where

$$
\begin{aligned}
\mathcal{J} &= \frac{1}{3} + \frac{4n+2}{144(n+2)}(2\pi)^{-2}\omega_n^{\frac{4}{n}}k^{-\frac{2}{n}} - 2(2\pi)^2 k^{\frac{2}{n}}\omega_n^{-\frac{4}{n}} \\
&< \frac{1}{3} + \frac{4(n+2)}{144(n+2)}(2\pi)^{-2}\omega_n^{\frac{4}{n}} - 2(2\pi)^2\omega_n^{-\frac{4}{n}} \\
&= \frac{1}{3} + \frac{1}{36}(2\pi)^{-2}\omega_n^{\frac{4}{n}} - 2(2\pi)^2\omega_n^{-\frac{4}{n}} \\
&< \frac{1}{3} + \frac{1}{72} - 4 \\
&< 0,
\end{aligned}
$$

which implies that $F'((2\pi)^{-n}Vol(D)) < 0$. Here, we use the inequality $\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} < \frac{1}{2}$. We can replace $\phi_2(0)$ by $(2\pi)^{-n}Vol(D)$ to obtain

$$
\begin{aligned}
\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq{}& \frac{nk^{\frac{2}{n}}}{n+2}\omega_n^{-\frac{2}{n}}(2\pi)^2 Vol(D)^{-\frac{2}{n}} + \frac{1}{24(n+2)}\frac{Vol(D)}{Ine(D)} \\
&+ \frac{nk^{-\frac{2}{n}}}{2304(n+2)^2}\omega_n^{\frac{2}{n}}(2\pi)^{-2}\left(\frac{Vol(D)}{Ine(D)}\right)^2 Vol(D)^{\frac{2}{n}}.
\end{aligned}
$$

since $\sigma = 2(2\pi)^{-n}\sqrt{Vol(D)Ine(D)}$.

This completes the proof of Theorem 1.1.

$\square$

Next, we will give the proof of Theorem 1.2.

Proof of Theorem 1.2.: Define the function $\phi_1(x)$ by $\phi_1(|x|) := f_1^*(x)$. Then we know that $\phi_1 : [0, +\infty) \to [0, (2\pi)^{-n}V(\Omega)]$ is a non-increasing function with respect to $|x|$. Taking

$$
b = n, \quad \psi(s) = \phi_1(s), \quad A = \frac{k}{n\omega_n}, \quad \text{and} \quad \mu = \sigma = 2(2\pi)^{-n}\sqrt{Vol(D)Ine(D)},
$$

we can obtain from Lemma 2.3 and (3.16) that

$$
\begin{aligned}
\sum_{j=1}^{k}\Lambda_j^{(\alpha)} \geq{}& n\omega_n\int_0^\infty s^{n+\alpha-1}\phi_1(s)ds \\
\geq{}& \frac{n\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha}{n}}}{n+\alpha}\phi_1(0)^{-\frac{\alpha}{n}} + \frac{\alpha\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha-2}{n}}}{12(n+\alpha)\sigma^2}\phi_1(0)^{\frac{2n-\alpha+2}{n}} \qquad (3.18) \\
&+ \frac{\alpha(n+\alpha-2)^2\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha-4}{n}}}{\mathcal{C}_1(n)n(n+\alpha)^2\sigma^4}\phi_1(0)^{\frac{4n-\alpha+4}{n}},
\end{aligned}
$$

where

$$
\mathcal{C}_1(n) = \begin{cases} 288, & \text{when } n \geq 4, \\ 384, & \text{when } n = 2 \text{ or } n = 3. \end{cases}
$$

Moreover, we define a function $\xi(t)$ by letting

$$
\begin{aligned}
\xi(t) &= \frac{n\omega_n}{n+\alpha}\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha}{n}} t^{-\frac{\alpha}{n}} + \frac{\alpha\omega_n}{12(n+\alpha)\sigma^2}\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha-2}{n}} t^{\frac{2n-\alpha+2}{n}} \\
&\quad + \frac{\alpha(n+\alpha-2)^2\omega_n}{\mathcal{C}_1(n)n(n+\alpha)^2\sigma^4}\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha-4}{n}} t^{\frac{4n-\alpha+4}{n}}.
\end{aligned}
\tag{3.19}
$$

Differentiating (3.19) with respect to the variable t, it is not difficult to see that

$$
\begin{aligned}
\xi'(t) &= \frac{\alpha\omega_n}{n+\alpha}\left(\frac{k}{\omega_n}\right)^{\frac{n+\alpha}{n}} t^{-\frac{\alpha}{n}-1}\left[-1 + \frac{(2n-\alpha+2)}{12n\sigma^2}\left(\frac{k}{\omega_n}\right)^{-\frac{2}{n}} t^{\frac{2n+2}{n}} \right. \\
&\quad \left. + \frac{(4n-\alpha+4)(n+\alpha-2)^2}{\mathcal{C}_1(n)n^2(n+\alpha)\sigma^4}\left(\frac{k}{\omega_n}\right)^{-\frac{4}{n}} t^{\frac{4n+4}{n}} \right].
\end{aligned}
\tag{3.20}
$$

Letting

$$
\zeta(t) = \xi'(t)\left(\frac{n+\alpha}{\alpha\omega_n}\right)\left(\frac{k}{\omega_n}\right)^{-\frac{n+\alpha}{n}} t^{\frac{\alpha}{n}+1},
\tag{3.21}
$$

and noting that $\sigma \geq (2\pi)^{-n}\omega_n^{-\frac{1}{n}} Vol(D)^{\frac{n+1}{n}}$, we can obtain from (3.20) and (3.21) that

$$
\begin{aligned}
\zeta(t) &= -1 + \frac{(2n-\alpha+2)}{12n\sigma^2}\left(\frac{k}{\omega_n}\right)^{-\frac{2}{n}} t^{\frac{2n+2}{n}} \\
&\quad + \frac{(4n-\alpha+4)(n+\alpha-2)^2}{\mathcal{C}_1(n)n^2(n+\alpha)\sigma^4}\left(\frac{k}{\omega_n}\right)^{-\frac{4}{n}} t^{\frac{4n+4}{n}} \\
&\leq -1 + \frac{(2n-\alpha+2)}{12n(2\pi)^{-2n}\omega_n^{-\frac{2}{n}} Vol(D)^{\frac{2(n+1)}{n}}}\left(\frac{k}{\omega_n}\right)^{-\frac{2}{n}} t^{\frac{2n+2}{n}} \\
&\quad + \frac{(4n-\alpha+4)(n+\alpha-2)^2}{\mathcal{C}_1(n)n^2(n+\alpha)(2\pi)^{-4n}\omega_n^{-\frac{4}{n}} Vol(D)^{\frac{4(n+1)}{n}}}\left(\frac{k}{\omega_n}\right)^{-\frac{4}{n}} t^{\frac{4n+4}{n}}.
\end{aligned}
\tag{3.22}
$$

It is easy to see that the right hand side of (3.22) is an increasing function of t. Therefore, if the right hand side of (3.22) is less than 0 when we take $t =$

$(2\pi)^{-n}Vol(D)$, which is equivalent to say that

$$
\begin{aligned}
\zeta(t) \leq &-1 + \frac{(2n - \alpha + 2)}{12n}k^{-\frac{2}{n}}\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2}\\
&+ \frac{(4n - \alpha + 4)(n + \alpha - 2)^2}{\mathcal{C}_1(n)n^2(n + \alpha)}k^{-\frac{4}{n}}\frac{\omega_n^{\frac{8}{n}}}{(2\pi)^4}\\
\leq &\ 0,
\end{aligned}
\tag{3.23}
$$

we can claim from (3.23) that $\xi'(t) \leq 0$ on $(0, (2\pi)^{-n}V(\Omega)]$. By a direct calculation, we can obtain

$$
\begin{aligned}
\zeta(t) &\leq -1 + \frac{(2n - \alpha + 2)}{12n} + \frac{(4n - \alpha + 4)(n + \alpha - 2)^2}{\mathcal{C}_1(n)n^2(n + \alpha)}\\
&\leq -1 + \frac{(2n + n)}{12n} + \frac{(4n + 2n)(n + n)^2}{\mathcal{C}_1(n)n^3}\\
&= -\frac{3}{4} + \frac{24}{\mathcal{C}_1(n)}\\
&\leq 0,
\end{aligned}
\tag{3.24}
$$

since $\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} < 1$. Thus, it is easy to see from (3.21) and (3.24) that $\xi'(t) \leq 0$, which implies that $\xi(t)$ is a decreasing function on $(0, (2\pi)^{-n}Vol(D)]$.

On the other hand, we notice that $0 < \phi_1(0) \leq (2\pi)^{-n}Vol(D)$ and right hand side of the formula (3.18) is $\xi(\phi_1(0))$, which is a decreasing function of $\phi_1(0)$ on $(0, (2\pi)^{-n}Vol(D)]$. Therefore, $\phi_1(0)$ can be replaced by $(2\pi)^{-n}Vol(D)$ in (2.1) which gives the following inequality:

$$
\begin{aligned}
\frac{1}{k}\sum_{j=1}^{k}\Lambda_j^{(\alpha)} \geq &\ \frac{n}{n + \alpha}\frac{(2\pi)^\alpha}{(\omega_n Vol(D))^{\frac{\alpha}{n}}}k^{\frac{\alpha}{n}}\\
&+ \frac{\alpha}{48(n + \alpha)}\frac{(2\pi)^{\alpha-2}}{(\omega_n Vol(D))^{\frac{\alpha-2}{n}}}\frac{Vol(D)}{Ine(D)}k^{\frac{\alpha-2}{n}}\\
&+ \frac{\alpha(n + \alpha - 2)^2}{\mathcal{C}(n)n(n + \alpha)^2}\frac{(2\pi)^{\alpha-4}}{(\omega_n Vol(D))^{\frac{\alpha-4}{n}}}\left(\frac{Vol(D)}{Ine(D)}\right)^2 k^{\frac{\alpha-4}{n}},
\end{aligned}
$$

where

$$
\mathcal{C}(n) = \begin{cases} 4608, & \text{when } n \geq 4,\\ 6144, & \text{when } n = 2 \text{ or } n = 3. \end{cases}
$$

In particular, when $\alpha = 2$, we can get the inequality (1.15) by using the same method as the proof of Theorem 1.1.

This completes the proof of Theorem 1.2.

$\square$

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